

INFINITE J -MATRICES AND A MATRIX MOMENT PROBLEM

M. KREIN

(Communicated by academician A. N. Kolmogorov, September 14, 1949)

In the paper [1], which deals with the theory of the representation of Hermitian operators with deficiency index (p, p) , we worked out as an illustration the application to the matrix moment problem.

The present paper includes several important supplements for the investigation of this problem. These supplements arise in an attempt to construct a complete and self-contained theory of infinite regular J_p -matrices ($p > 1$) and, just as J_1 -matrices are related to the classical moment problem, this theory of J_p -matrices is related to the matrix moment problem. For clarity, this analogy led to a natural matrix notation, which considerably facilitates the subsequent analogy and facts from the theory of J_p -matrices and the matrix moment problem.

We say that the infinite Hermitian matrix $\mathbf{A} = (a_{i,k})_0^\infty$ is a *regular J_p -matrix* when it can be written as $\mathbf{A} = (A_{i,k})_0^\infty$, where $A_{i,k}$ ($i, k = 0, 1, 2, \dots$) are square $p \times p$ matrices for which $A_{i,k} = 0$ if $|i - k| > 1$ and $A_{i,i+1}$ ($i = 0, 1, 2, \dots$) are nonsingular. In particular, the Hermitian matrix $\mathbf{A} = (a_{i,k})_0^\infty$ is a regular J_p -matrix when $a_{i,k} = 0$ for $|i - k| > 1$ ¹ and $a_{i,i+p} \neq 0$ for $i = 0, 1, 2, \dots$.

Our theory of regular J_p -matrices is in our opinion of interest since it may serve as an algebraic model for one-dimensional boundary value problems with one singularity at an endpoint in a space of functions or vector functions.

Finally we note that infinite regular J_p -matrices for $p > 1$ have already been studied by H. Nagel [2]. However the deepest facts of the theory of this class of matrices have not been investigated by him.

1. Let $\mathbf{A} = (A_{i,k})_0^\infty$ be a regular J_p -matrix ($p > 1$). Denote by \mathfrak{M}_p the set of all square $p \times p$ matrices with complex entries and by \mathfrak{B} the collection of all polynomials $P(\lambda)$ given by

$$P(\lambda) = C_0 + C_1\lambda + \dots + C_n\lambda^n,$$

where $C_i \in \mathfrak{M}_p$ ($i = 0, 1, \dots, n$) and n may take any value from $0, 1, 2, \dots$.

We associate with the matrix \mathbf{A} the sequence of polynomials $D_k(\lambda) \in \mathfrak{B}$ ($k = 0, 1, 2, \dots$) defined by the recurrence relation

$$(1) \quad A_{k,k-1}D_{k-1}(\lambda) + (A_{k,k} - \lambda I)D_k(\lambda) + A_{k,k+1}D_{k+1}(\lambda) = 0$$

$$(k = 0, 1, \dots; D_{-1} \equiv 0),$$

Dokl. Akad. Nauk SSSR **69** nr. 2 (1949), 125–128.

Received 14 IX 1949

¹translator's note: presumably $|i - k| > p$ is meant

with the polynomial $D_0(\lambda)$ an arbitrary constant nonsingular matrix.

The condition that all the matrices $A_{i,i+1}$ are nonsingular implies that, for D_0 given, all matrix polynomials $D_k(\lambda)$ ($k = 1, 2, \dots$) are uniquely defined and moreover the polynomial $D_k(\lambda)$ has exact degree k with a nondegenerate matrix coefficient for λ^k ($k = 0, 1, 2, \dots$).

It is easy to see that for each complex number z the limit matrix

$$H(z) = \lim_{n \rightarrow \infty} \left(\sum_{k=0}^n D_k^*(\bar{z}) D_k(z) \right)^{-1}$$

exists.

Here and in what follows $P^*(\lambda)$ denotes the polynomial obtained from $P(\lambda) \in \mathfrak{B}$ by replacing each of its matrix coefficients by its Hermitian conjugate, so that

$$P^*(\bar{z}) = [P(z)]^*.$$

The following results holds (e.g., [2]).

Theorem 1. *The rank $r(z)$ of the Hermitian matrix $H(z)$ is the same for each z belonging to the same half-plane $\text{Im } z > 0$ or $\text{Im } z < 0$.*

We denote by ν_+ and ν_- the value of the rank $r(z)$ corresponding to respectively $\text{Im } z > 0$ and $\text{Im } z < 0$. The J_p -matrix \mathbf{A} , just as a Q -matrix, corresponds to an Hermitian operator (which we denote by the same letter \mathbf{A}) in a Hilbert space ℓ^2 consisting of $x = \{\xi_k\}_0^\infty$ with complex numbers ξ_k which are absolutely square summable.

Theorem 1 immediately leads to the following:

Theorem 2. *The numbers ν_+ and ν_- correspond to the upper and the lower deficiency index of the Hermitian operator \mathbf{A} .*

In the case of a real matrix \mathbf{A} it is clear that $\nu_+ = \nu_-$.

2. Every polynomial $P(\lambda) \in \mathfrak{B}$ of degree n can be written as

$$(2) \quad P(\lambda) = \sum_0^n U_k D_k(\lambda),$$

where $U_k \in \mathfrak{M}_p$ ($k = 1, 2, \dots, n$)².

For any $P, Q \in \mathfrak{B}$, where P is given by (2) and

$$Q(\lambda) = \sum_0^m V_k D_k(\lambda),$$

we put

$$\{P, Q\} = \sum_0^s U_k V_k^*, \quad (s = \min(n, m)).$$

²translator's note: $k = 0$ should of course also be included

In particular

$$(3) \quad \{D_i, D_j\} = \delta_{i,k} I \quad (i, k = 0, 1, 2, \dots).$$

The “form” $\{P, Q\}$ is completely defined by condition (3) and the following properties:

$$\{P_1 + P_2, Q\} = \{P_1, Q\} + \{P_2, Q\}; \quad \{P, Q_1 + Q_2\} = \{P, Q_1\} + \{P, Q_2\};$$

$$(4) \quad \{CP, Q\} = C\{P, Q\}; \quad \{P, CQ\} = \{P, Q\}C^*,$$

where C is any matrix in \mathfrak{M}_p .

It is easy to verify and well-known that by using the relations (3) the form $\{P, Q\}$ also has the following property:

$$\{\lambda P, Q\} = \{P, \lambda Q\} \quad (P, Q \in \mathfrak{B}).$$

Let us now form the sequence of matrices

$$S_n = \{\lambda^n I, I\} \quad (n = 0, 1, 2, \dots).$$

For each $X_j \in \mathfrak{M}_p$ ($j = 0, 1, 2, \dots$) one has

$$\sum_{j,k=0}^n X_j S_{j+k} X_k^* = \left\{ \sum_0^n X_j \lambda^j, \sum_0^n X_j \lambda^j \right\}.$$

On the other hand, for each $P \in \mathfrak{B}$ the expression $\{P, P\}$ is an Hermitian matrix (different from zero if $P(\lambda) \not\equiv 0$) corresponding to a non-negative definite form. Therefore by choosing matrices X_j ($j = 0, 1, 2, \dots$) for which all the rows, except the first one, consist of only zeros, we see that for arbitrary p -dimensional vectors $x_j = (\xi_{j,1}, \xi_{j,2}, \dots, \xi_{j,p})$, not identically zero, one has

$$(5) \quad \sum_{j,k=0}^n x_j S_{j+k} x_k^* > 0 \quad (n = 0, 1, 2, \dots).$$

It is easy to show that conversely if some sequence of matrices $\{S_n\}_0^\infty \subset \mathfrak{M}_p$ satisfies the condition (5) then it is induced by some regular J_p -matrix.

On the other hand, as has already been shown earlier [1;3], condition (5) is a necessary and sufficient condition for the solvability of the matrix moment problem

$$(6) \quad S_n = \int_{-\infty}^{\infty} \lambda^n dT(\lambda) \quad (n = 0, 1, 2, \dots),$$

where it is required to find an Hermitian matrix function $T(\lambda)$ subject to the condition that for each $x = (\xi_1, \xi_2, \dots, \xi_p) \neq 0$ the form $xT(\lambda)x^*$ is a nondecreasing function of $\lambda \in (-\infty, \infty)$ with an infinite number of points of increase.

In this way every regular J_p -matrix corresponds to some matrix moment problem. Our previous investigation [1] of the problem (6) makes it possible, in particular, to state the following:

Theorem 3. *The moment problem (6) has a unique normalized solution $T(\lambda)$ if and only if one of the numbers ν_+ or ν_- is equal to zero.*

We say that a solution $T(\lambda)$ of the moment problem is *normalized* if

$$\lim_{\lambda \rightarrow -\infty} T(\lambda) = 0, \quad T(\lambda - 0) = T(\lambda) \quad (-\infty < \lambda < \infty).$$

By the norm $\|C\|$ of a matrix $C \in \mathfrak{M}_p$ we mean the smallest number $\mu \geq 0$ with the property $C^*C \leq \mu^2 I$ (the inequality for Hermitian matrices is to be understood as an inequality for the corresponding forms).

Theorem 4. *If $\nu_+ = \nu_- = p$ (the case of a completely indeterminate moment problem (6)), then the series*

$$H^{-1}(z) = \sum_0^\infty D_k^*(\bar{z})D_k(z)$$

*converges uniformly on each bounded set of the complex plane and*³

$$\lim_{|z| \rightarrow \infty} \frac{\log \|H^{-1}(z)\|}{\|z\|} = 0.$$

Every solution $T(\lambda)$ of the moment problem (6) satisfies the inequality

$$T(\xi + 0) - T(\xi - 0) \leq H^{-1}(\xi) \quad (-\infty < \xi < \infty).$$

For a fixed ξ the equality sign is attained for one and only one normalized solution $T(\lambda) = T_\xi(\lambda)$ which is determined by the relation

$$(z - \xi)^{-1} \left[I + (z - \xi) \sum_1^\infty E_k^*(z)D_k(\xi) \right] \left[\sum_0^\infty D_k^*(z)D_k(\xi) \right]^{-1} = \int_{-\infty}^\infty \frac{dT_\xi(\lambda)}{\lambda - z} \quad (\text{Im } z < 0).$$

Here

$$E_k(z) = \left\{ \frac{D_k(\lambda) - D_k(z)}{\lambda - z}, I \right\} = \int_{-\infty}^\infty \frac{D_k(\lambda) - D_k(z)}{\lambda - z} dT(\lambda).$$

If $\nu_+ = \nu_- = p$, then the series

$$\sum_1^\infty E_k^*(z)D_k(\zeta), \quad \sum_1^\infty E_k^*(z)E_k(\zeta)$$

converge uniformly in the variables z and ζ on every bounded set of the complex plane.

Let us construct the entire matrix functions

$$F_1(z) = I + z \sum_1^\infty E_k^*(z)D_k(0), \quad F_2(z) = z \sum_1^\infty E_k^*(z)E_k(0),$$

$$G_1(z) = -z \sum_0^\infty D_k^*(z)D_k(0), \quad G_2(z) = I - z \sum_1^\infty D_k^*(z)E_k(0).$$

³translator's note: presumably the limit for $\|z\| \rightarrow \infty$ is meant

Theorem 5. *If $\nu_+ = \nu_- = p$, then each normalized solution $T(\lambda)$ of the problem (6) is in a one-to-one correspondence with the collection of all holomorphic $p \times p$ -matrix functions in the upper half-plane $V(z)$ with $\|V(z)\| \leq 1$ ($\text{Im } z > 0$), such that*

$$\begin{aligned} [F_1(z)(I + V(z)) + iF_2(z)(I - V(z))] [G_1(z)(I + V(z)) \\ + iG_2(z)(I - V(z))]^{-1} = \int_{-\infty}^{\infty} \frac{dT(\lambda)}{\lambda - z}. \end{aligned}$$

We do not turn our attention here to the connection which exists between the (usual and generalized) resolvent of the operator \mathbf{A} and the solution $T(\lambda)$ of problem (6), which essentially has been explained in our paper [1]. We only note that because of this connection each self-adjoint extension $\tilde{\mathbf{A}}$ of the operator \mathbf{A} corresponds to a certain unitary matrix U (and vice versa) in such a way that the spectrum of the operator $\tilde{\mathbf{A}}$ coincides with the set of roots of the equation $\det [G_1(z)(I + U) + iG_2(z)(I - U)] = 0$.

REFERENCES

1. M. Krein, *Fundamental aspects of the representation theory of Hermitian operators with deficiency index (m, m)* , Ukrain. Mat. Ž. **1** (1949), 3–66; English transl. in Amer. Math. Soc. Transl. (2) **97** (1970), 75–143.
2. H. Nagel, *Über die aus quadrierbaren Hermiteschen Matrizen entstehenden Operatoren*, Math. Ann. **112** **109** (1936), 247–285.
3. M. Krein and M. Krasnoselskiĭ, *Fundamental theorems on the extension of Hermitian operators and some applications to the theory of orthogonal polynomials and the moment problem*, Uspehi Mat. Nauk **2** (1947), nr. 3 (19), 60–106.

Translated by W. VAN ASSCHE